New families of small regular graphs of girth 5

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Abstract

In this paper we are interested in the *Cage Problem* that consists in constructing regular graphs of given girth g and minimum order. We focus on girth $g = 5$, where cages are known only for degrees $k \leq 7$. We construct regular graphs of girth 5 using techniques exposed by Funk [Note di Matematica. 29 suppl.1, (2009) 91 - 114] and Abreu et al. [Discrete Math. 312 (2012), 2832 - 2842] to obtain the best upper bounds known hitherto. The tables given in the introduction show the improvements obtained with our results.

Keyword 1 *Small regular graphs, cage, girth, amalgam.*
AMS subject classification: *05C35, 05C38.*

1 Introduction

All the graphs considered are finite and simple. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the size $g = g(G)$ of a shortest cycle. The degree of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called k-regular if all its vertices have the same degree k, and bi-regular or (k_1, k_2)-regular if all its vertices have either degree k_1 or k_2. A (k, g)-graph is a k-regular graph with girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices; the order of a (k, g)-cage is denoted by $n(k, g)$. Cages were introduced by Tutte [30] in 1947 and their existence was proved by Erdős and Sachs [14] in 1963 for any values of regularity and girth. The lower bound on the number of

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vertices of a (k,g)-graph is denoted by $n_0(k,g)$, and it is calculated using the distance partition with respect either a vertex (for odd g), or and edge (for even g):

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

Obviously a graph that attains this lower bound is a (k,g)-cage. Biggs [11] calls *excess* of a (k,g)–graph G the difference $|V(G)| - n_0(k,g)$. There has been intense work related with *The Cage Problem*, focussed on constructing the smallest (k,g)-graphs (for a complete survey of this topic see [16]).

In this paper we are interested in the cage problem for $g = 5$, in this case $n_0(k,5) = 1 + k^2$. It is well known that this bound is attained for $k = 2, 3, 7$ and perhaps for $k = 57$ (see [11]) and that for $k = 4, 5, 6$, the known graphs of minimum order are cages (see [22, 23, 24, 25, 29, 31, 32, 33]).

Jørgensen [20] establishes that $n(k,5) \leq 2(q-1)(k-2)$ for every odd prime power $q \geq 13$ and $k \leq q+3$. Abreu et al. [1] prove that $n(k,5) \leq 2(qk-3q-1)$ for any prime $q \geq 13$ and $k \leq q+3$, improving Jorgensen's bound except for $k = q+3$ where both coincide.

In [17] Funk uses a technique that consists in finding regular graphs of girth greater or equal than five and performing some operations of amalgams and reductions in the (bipartite) incidence graph, also called Levi Graph of elliptic semiplanes of type C and L (see [5, 13, 17]). In this paper we improve some results of Funk finding the best possible regular graphs to amalgamate which allows us to obtain new better upper bounds. To do that, we also use the techniques given in [1, 2] where the authors not only amalgamate regular graphs, but also bi-regular graphs. In this paper new $(k,5)$-graphs are constructed for $17 \leq k \leq 22$ using the incidence graphs of elliptic semiplanes of type C. The new upper bounds appear in the last column of Table 1, which also shows the current values for $8 \leq k \leq 22$. To evaluate our achievements, we follow the notation in [16, 17], and let $rec(k,5)$ denote the smallest currently known order of a k-regular graph of girth 5. Hence $n(k,5) \leq rec(k,5)$.

k	$rec(k,5)$	Due to	Reference	New $rec(k,5)$
8	80	Royle, Jørgensen	[26, 20]	
9	96	Jørgensen	[20]	
10	124	Exoo	[15]	
11	154	Exoo	[15]	
12	203	Exoo	[15]	
13	230	Exoo	[15]	
14	284	Abreu et al.	[1]	
15	310	Abreu et al.	[1]	
16	336	Jørgensen	[20]	
17	448	Schwenk	[27]	*436*
18	480	Schwenk	[27]	*468*
19	512	Schwenk	[27]	*500*
20	572	Abreu et al.	[1]	*564*
21	682	Abreu et al.	[1]	*666*
22	720	Jørgensen	[20]	*704*

Table 1: Current and our new values of $rec(k,5)$ for $8 \leq k \leq 22$.

The bounds obtained by Funk using elliptic semiplanes of type L on $n(k,5)$ for $23 \leq k \leq 31$

remain untouched whereas, for $32 \leq k \leq 52$, we obtain the best possible regular graphs to amalgamate in this type of incidence graphs obtaining, in consequence, the best upper bounds known hitherto (see Table 2).

k	$rec(k,5)$	Due to	Reference	New $rec(k,5)$
32	1680	Jørgensen	[20]	*1624*
33	1856	Funk	[17]	*1680*
34	1920	Jørgensen	[20]	*1800*
35	1984	Funk	[17]	*1860*
36	2048	Funk	[17]	*1920*
37	2514	Abreu et al	[1]	*2048*
38	2588	Abreu et al	[1]	*2448*
39	2662	Abreu et al	[1]	*2520*
40	2736	Jørgensen	[20]	*2592*
41	3114	Abreu et al	[1]	*2664*
42	3196	Abreu et al	[1]	*2736*
43	3278	Abreu et al	[1]	*3040*
44	3360	Jørgensen	[20]	*3120*
45	3610	Abreu et al	[1]	*3200*
46	3696	Jørgensen	[20]	*3280*
47	4134	Abreu et al	[1]	*3360*
48	4228	Abreu et al	[1]	*3696*
49	4322	Abreu et al	[1]	*4140*
50	4416	Jørgensen	[20]	*4232*
51	4704	Jørgensen	[20]	*4324*
52	4800	Jørgensen	[20]	*4416*

Table 2: Current and our new values of $rec(k,5)$ for $32 \leq k \leq 52$.

Finally, when $q \geq 49$ is a prime power, the search for 6-regular suitable pairs of graphs has allowed us to establish the two following general results. Note that the bounds are different depending on the parity of q.

Theorem 1.1 *Given an integer $k \geq 53$, let q be the lowest odd prime power, such that $k \leq q+6$. Then $n(k,5) \leq 2(q-1)(k-5)$.*

Theorem 1.2 *Given an integer $k \geq 68$, let $q = 2^m$ be the lowest even prime, such that $k \leq q+6$. Then $n(k,5) \leq 2q(k-6)$.*

Since the bounds of Theorem 1.1 and Theorem 1.2, associated to primes $q = 49$ and $q = 64$, represent a considerable improvement to the current known ones, we give them explicitly.

k	$rec(k,5)$	Due to	Reference	New $rec(k,5)$
55	5510	Abreu et al	[1]	*4800*
70	8976	Jørgensen	[20]	*8192*

To finalize the introduction we want to empathize that Funk, in [17], gives a pair of 4-regular graphs of girth 5 suitable for amalgamation in some specific incidence graphs of elliptic semi-planes and he posed the question about the existence of a pair of 5-regular graphs with the

same objective (obviously these graphs should have the same order that those given by Funk and also girth 5), in this paper we exhibit the graphs which solve this problem. Furthermore, let us notice that all the bounds on $n(k,5)$ contained in this paper are obtained constructively, that is, for each k, we construct a $(k,5)$-graph with order *new* $rec(k,5)$.

2 Preliminaries

A useful tool to construct k-regular graphs of girth 5 is the operation of *amalgamation* on the incidence graph of an elliptic semiplane (Jørgensen [20] 2005 and Funk [17] 2009).

For $q = p^m$ a prime power, consider the Levi graphs C_q and L_q of the so-called elliptic semiplanes of type C and L, respectively. Recall that the semiplane of type C is obtained from the projective plane over the field $\mathbb{F}_q$ by deleting a point and all the lines incident with it together with all the points that belong to one of these lines, and that the Levi graph C_q is bipartite, q-regular and has $2q^2$ vertices which corresponds in the elliptic semiplane to q^2 points and q^2 lines both partitioned into q parallel classes or blocks of q elements each. On the other hand, the semiplane of type L is obtained by deleting from the projective plane a point and all the lines incident with it together with a different line and all its points, here the Levi graph of L_q is also bipartite, q-regular and has $2(q^2-1)$ vertices of which q^2-1 are points and q^2-1 are lines in the elliptic semiplane, both partitioned into $q+1$ parallel classes of $q-1$ elements each.

The construction of our new graphs consists in finding regular and bi-regular graphs of girth greater or equal than five and performing some operations of amalgams and reductions in C_q or L_q. In [20], Jørgensen exploits these ideas and proves that two graphs are suitable for amalgamation (one of them in each block of points and the other in each block of lines) if they have disjoint sets of Cayley colors.

In [1] these ideas are also used to construct graphs using the elliptic semiplane of type C, and the main Theorem of [1] was refined in [2] to construct bi-regular cages of girth 5. In fact, the suitable graphs to amalgamate can have some common Cayley color, but only for some specific edges.

The paper is organized as follows. In the next Section we work with elliptic semiplanes of type C and with techniques used in [1, 2], in Section 4 with elliptic semiplanes of type L and with techniques given by Funk in [17]. Finally, in Section 5 we return to the elliptic semiplanes of type C for even primes because they require new descriptions.

3 Amalgamating into elliptic semiplanes of type C

Let q be a prime power and $\mathbb{F}_q$ the finite field of order q; we recall the definition and properties of the incidence bipartite graph C_q of an elliptic semiplane of type C exactly as they appear in [1, 2]. Notice that in these papers the authors call this graph B_q and here, as it is related to the elliptic semiplane of type C, we prefer to call it C_q.

Definition 3.1 *Let C_q be a bipartite graph with vertex set (V_0, V_1) where $V_r = \mathbb{F}_q \times \mathbb{F}_q$, $r = 0, 1$; and the edge set defined as follows:*

$$(x, y)_0 \in V_0 \text{ adjacent to } (m, b)_1 \in V_1 \text{ if and only if } y = mx + b. \tag{1}$$

The graph C_q is also known as the incidence graph of the biaffine plane [18] and it has been used in the problem of finding extremal graphs without short cycles (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 21]). The graph C_q is q-regular of order $2q^2$, has girth $g = 6$ for $q \geq 3$ and it is vertex transitive. Other properties of the graph C_q are well known (see [1, 2, 18, 21]).

Let Γ_1 and Γ_2 be two graphs of the same order and with the same labels on their vertices; an *amalgam of* Γ_1 *into* Γ_2 is a graph obtained adding all the edges of Γ_1 into Γ_2. In [1] it is described a technique of amalgamation of two r-regular graphs H_0, H_1 and two $(r, r+1)$-regular graphs G_0, G_1 (all of them of girth at least 5 and with some specific properties) into a subgraph of C_q obtaining the resulting amalgam graph, denoted by $\mathcal{C}_q(H_0, H_1, G_0, G_1)$, which is $(q+r)$-regular and has girth at least five.

Theorem 3.1 is a reformulation of Theorem 5 in [1] (with a new strong hypothesis that also appears in Theorem 4.9 in [2]). Recall that if G is a graph with $V(G)$ labeled with the elements of $\mathbb{F}_q$ and $\alpha\beta$ is an edge of G, then the *Cayley Color* or *weight* of $\alpha\beta$ is $\pm(\alpha - \beta) \in \mathbb{F}_q - \{0\}$.

Theorem 3.1 *Let $q \geq 3$ be a prime power and $r \geq 2$ an integer. Consider graphs H_0, H_1, G_0 and G_1 with the following properties:*

(i) *$V(G_i) = \mathbb{F}_q$ and G_i is an $(r, r+1)$-regular graph of girth $g(G_i) \geq 5$ for $i = 0, 1$;*

(ii) *H_i is an r-regular graph of girth $g(H_i) \geq 5$ and $V(H_i) = \{v \in \mathbb{F}_q : d_{G_j}(v) = r \text{ with } i \neq j\}$, for $i, j \in \{0, 1\}$.*

(iii) *$E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$, $E(H_1) \cap E(G_0) = \emptyset$ and G_0 and G_1 have disjoint Cayley colors.*

Consider the sets of vertices $P'_0 = \{(0, y)_0 : y \in V(H_0)\}$, $L'_0 = \{(0, b)_1 : b \in V(H_1)\}$; for all $x, m \in \mathbb{F}_q - 0$, let $P_x = \{(x, y)_0 : y \in \mathbb{F}_q\}$ and $L_m = \{(m, b)_1 : b \in \mathbb{F}_q\}$. Let $A = P'_0 \cup (\bigcup_{x \in \mathbb{F}_q - 0} P_x) \cup L'_0 \cup (\bigcup_{m \in \mathbb{F}_q - 0} L_m)$ and consider the induced subgraph $\mathcal{C}_q[A]$ where $\mathcal{C}_q$ is the graph given in Definition 1.

Moreover, let the sets of edges $E_0(0) = \{(0, y)_0(0, y')_0 : yy' \in E(H_0)\}$, $E_1(0) = \{(0, b)_1(0, b')_1 : bb' \in E(H_1)\}$, $E_0(x) = \{(x, y)_0(x, y')_0 : yy' \in E(G_0)\}$, $E_1(m) = \{(m, b)_1(m, b')_1 : bb' \in E(G_1)\}$ for all $m, x \in \mathbb{F}_q - 0$.

The graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ with vertex set A and edge set $E(C_q[A]) \cup (\bigcup_{x \in \mathbb{F}_q} E_0(x)) \cup (\bigcup_{m \in \mathbb{F}_q} E_1(m))$ is $(q+r)$-regular and has girth at least five.

The proof is the same as the one of Theorem 4.9 in [2]. Notice that Theorem 3.1 can also be applied when G_0 and G_1 are regular graphs (then $H_0 = G_0$ and $H_1 = G_1$). In this case we denote the resulting graph by $\mathcal{C}_q(G_0, G_1)$.

Next, for primes $q \in \{16, 17, 19\}$, we construct graphs H_0, H_1, G_0, G_1, which verify the conditions of Theorem 3.1.

Construction 1:

- For $q = 16$:

  Let $(\mathbb{F}_{16}, +) \cong ((\mathbb{Z}_2)^4, +)$ be a finite field of order 16 with set of elements $\{(d, e, f, g) : d, e, f, g \in \mathbb{Z}_2\}$, we write $defg$ instead of (d, e, f, g). Consider the graphs H_0, H_1, G_0 and G_1 displayed in Figure 1. The graphs G_0 and G_1 are not isomorphic, although both have girth 5 and order 16, with 6 vertices of degree 4 and 10 vertices of degree 3. The labeling of the vertices of G_0 and G_1 is such that the vertices of the set $S = \{0000, 1100, 0110, 1001, 0011, 1111\}$ have degree four and the other ones have degree three. The weights or Cayley colors of G_0 (and G_1) are $\{0001, 0010, 0100, 1000, 1111\}$ (and $\{0011, 0110, 0111, 1001, 1010, 1011, 1100, 1101, 1110\}$, respectively). Hence, G_0 and G_1 have disjoint sets of Cayley colors. Moreover, the graphs H_0 and H_1 are isomorphic to the Petersen graph and they are labeled with the elements of $(\mathbb{Z}_2)^4 - S$ in such a way that $E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$ and $E(H_1) \cap E(G_0) = \emptyset$.

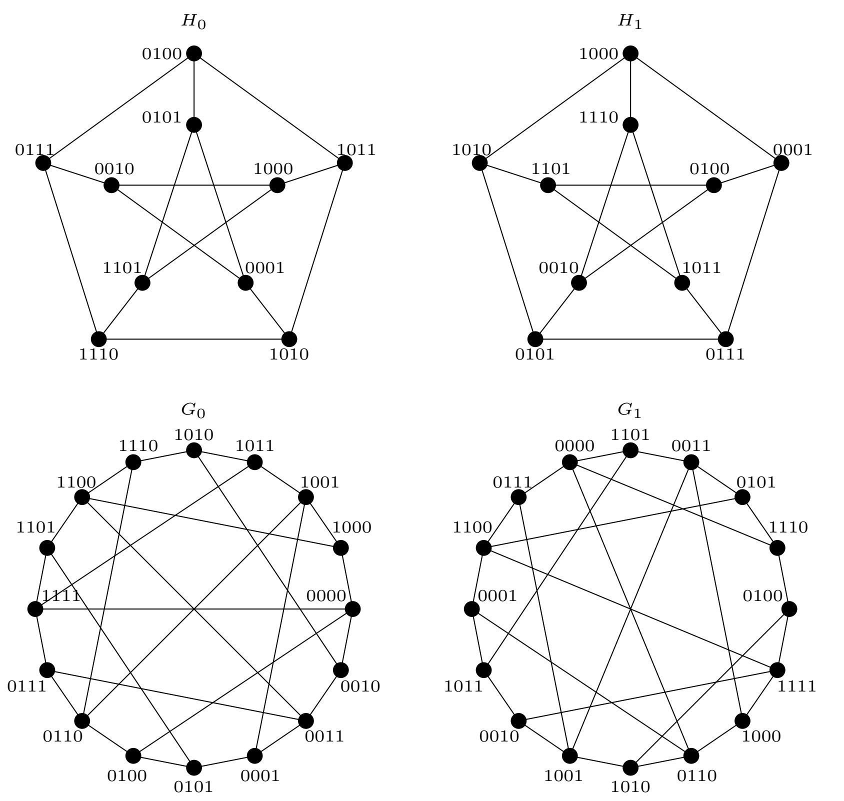

Figure 1: The graphs H_i and G_i for $i = 0, 1$ where $q = 16$.

- For $q = 17$:

  Let $\mathbb{Z}_{17}$ be a finite field of order 17 and consider the graphs H_0, H_1, G_0 and G_1 displayed in Figure 2. The graphs G_0 and G_1 are isomorphic, with 7 vertices of degree 4 and 10

vertices of degree 3. Both graphs have the same set of vertices $S = \{0, 2, 5, 8, 10, 13, 15\}$ of degree 4, and the Cayley colors of G_0 (and G_1) are $\pm\{1, 5, 8\}$ (and $\pm\{2, 3, 4, 6, 7\}$, respectively). Regarding H_0 and H_1, they are labeled with the elements of $\mathbb{Z}_{17} - S$ and verify $E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$ and $E(H_1) \cap E(G_0) = \emptyset$.

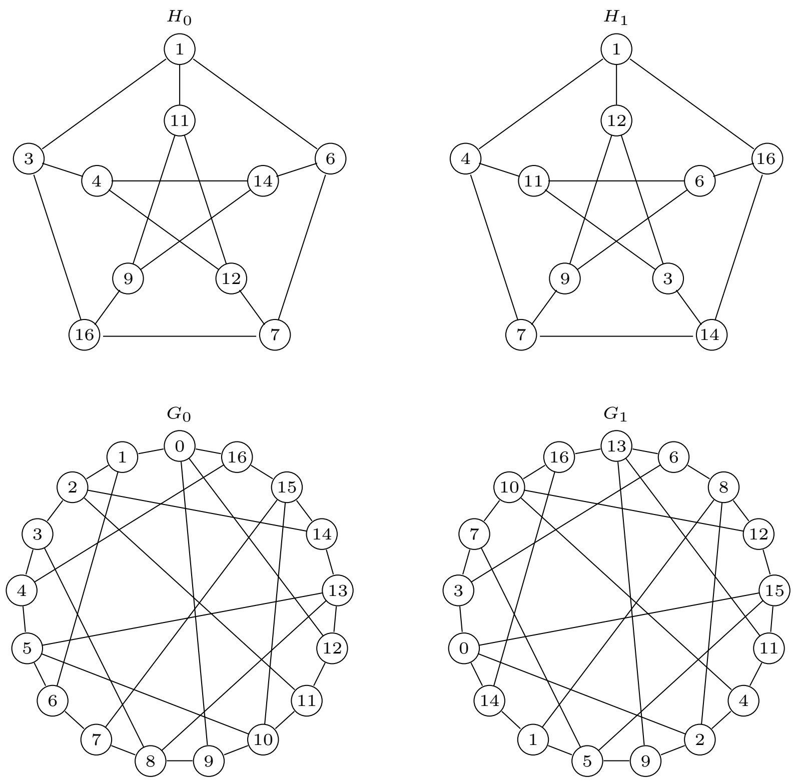

Figure 2: The graphs H_i and G_i for $i = 0, 1$ where $q = 17$.

- For $q = 19$:

  Let $\mathbb{Z}_{19}$ be a finite field of order 19 and consider the graphs H_0, H_1, G_0 and G_1 showed in Figure 3. The graphs G_0 and G_1 are isomorphic, have order 19, girth 5, the vertices of the set $S = \{0, 2, 3, 5, 6, 12, 13, 16, 17\}$ have degree 4 and the other ones have degree 3. The weights or Cayley colors modulo 19 of G_0 (and G_1) are $\pm\{1, 4, 7, 8\}$ (and $\pm\{2, 3, 5, 6, 9\}$, respectively). Regarding H_0 and H_1, they are labeled with the elements of $\mathbb{Z}_{19} - S$ and verify $E(H_0) \cap E(H_1) = \emptyset$, $E(H_0) \cap E(G_1) = \emptyset$ and $E(H_1) \cap E(G_0) = \emptyset$.

In the next result we apply Theorem 3.1 to $q \in \{16, 17, 19\}$. The obtained graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ is a $(q+3, 5)$-regular graph with less vertices than any other $(q+3)$-regular graph of girth 5 so far known, and therefore the upper bound $rec(k, 5)$ for $k \in \{19, 20, 22\}$ is improved. As it is explained in the so-called Reduction 2 in [1], referred as "Deletion" in [17], by removing pairs of blocks P_x and L_m from $\mathcal{C}_q(H_0, H_1, G_0, G_1)$, we also generate new graphs which improve $rec(k, 5)$ for $k \in \{17, 18, 21\}$.

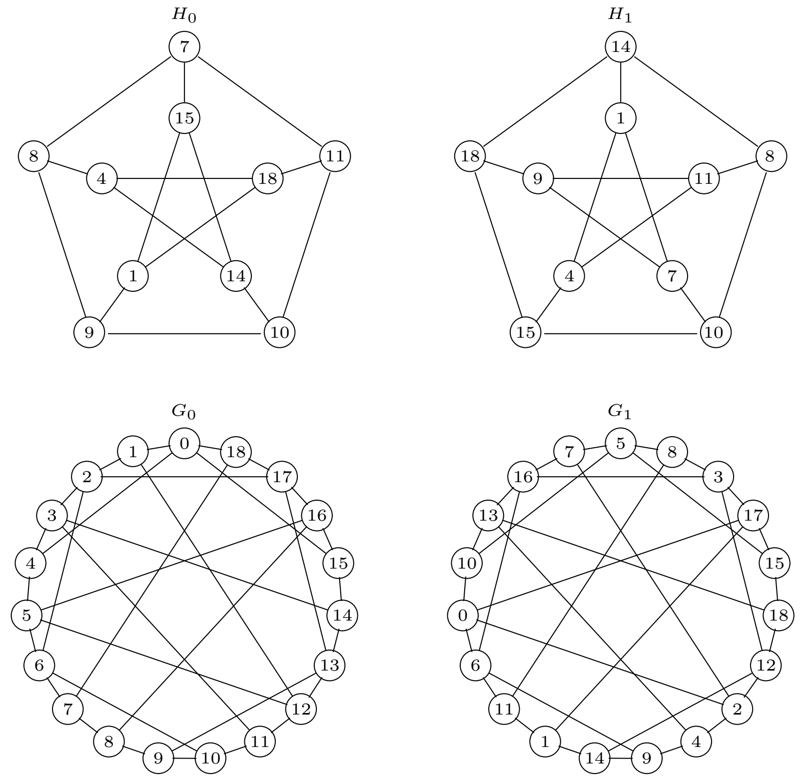

Figure 3: The graphs H_i and G_i for $i = 0, 1$ where $q = 19$.

Theorem 3.2 *The following upper bound $rec(k, 5)$ on the order $n(k, 5)$ of a k-regular cage of girth 5 holds*

k	$rec(k,5)$
17	*436*
18	*468*
19	*500*
20	*564*
21	*666*
22	*704*

Proof Using the graphs given in Construction 1, we obtain for $q \in \{16, 17, 19\}$ the graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ as in Theorem 3.1, which has girth 5. Moreover we have the following considerations:

For $q = 16$, $\mathcal{C}_{16}(H_0, H_1, G_0, G_1)$ is a $(19, 5)$-graph of order $2 \cdot 16^2 - 12 = 500$ implying that any $(19, 5)$-cage has at most 500 vertices. Removing of $\mathcal{C}_{16}(H_0, H_1, G_0, G_1)$ (using the operation called "Reduction 2" in [1]) a block of lines L_m and a block of points P_x, for $x, m \in (\mathbb{Z}_2)^4 - \{0000\}$, we construct a 18-regular graph with $500 - 2 \cdot 16 = 468$ vertices. Similarly, deleting from this last graph another pair of blocks we obtain a 17-regular graph of girth 5 with 436 vertices. Each of these k-regular graphs ($k = 17, 18, 19$) has 12 vertices less than the ones constructed by Schwenk in [27].

For $q = 17$, $\mathcal{C}_{17}(H_0, H_1, G_0, G_1)$ is a $(20, 5)$-graph of order $2 \cdot 17^2 - 14 = 564$, which implies that a $(20, 5)$-cage has at most 564 vertices.

For $q = 19$, $\mathcal{C}_{19}(H_0, H_1, G_0, G_1)$ is a $(22, 5)$-graph of order $2 \cdot 19^2 - 18 = 704$, which also implies that any $(22, 5)$-cage has at most 704 vertices. Newly, deleting any block of points and any block of lines (except P_0 and L_0 blocks), it is straightforward to check out that $n(21, 5) \leq 666$. ■

Remark 3.1 *It is important to note that the construction of a $(q+3)$-regular graph of girth at least 5 using bi-regular amalgams into a subgraph of C_q involves the existence of two 3-regular graphs H_0 and H_1 and two $(3, 4)$-regular graphs G_0 and G_1 all of them with girth at least 5. The graph $\mathcal{C}_q(H_0, H_1, G_0, G_1)$ has order $2(q^2 - (q - n(H_0)) \geq 2(q^2 - q + 10)$. It means that our construction is the best possible one for $q = 16$ and $q = 17$, because a 4-regular amalgam could only be possible for $q \geq n(4, 5) = 19$ (recall that the $(4, 5)$-cage is the Robertson Graph that has order 19).*

4 Amalgamating into elliptic semiplanes of type L

In this section we use the techniques given by Funk in [17] to amalgamate a pair of suitable regular graphs into the Levi graph of an elliptic semiplane of type L_q. Recall that the semiplane of type L is obtained by deleting, from the projective, a pair of non incident point and line, all the lines incident with the point and all the points incident with the line. Moreover, the Levi graph, denoted by L_q, is bipartite, q-regular and has $2(q^2 - 1)$ vertices of which $q^2 - 1$ are points and $q^2 - 1$ are lines in the elliptic semiplane, both partitioned into $q + 1$ parallel classes of $q - 1$ elements each.

We divide the section into two parts. First we construct the regular graphs G_0, G_1 to amalgamate and later we describe the resulting graph $\mathcal{L}_q(G_0, G_1)$.

4.1 Constructions of regular graphs of girth five

To apply the Funk's techniques we need to construct two regular graphs with the same order, girth at least five and having disjoint Cayley colors, one of them to amalgamate in the point blocks and the other in the line blocks of L_q.

Let $\mathbb{Z}_n$ be the set of integers modulo n, and $J = \{k_1, \ldots, k_w\} \subset \mathbb{Z}_n - 0$. Recall that a *circulant graph* $Z_n(J)$ is a graph with vertex set $\mathbb{Z}_n$ and edges $\alpha\beta$ where $\beta - \alpha \in J$. Let $n = 2t$ and suppose that every element of J is odd. We denote by $S_{2t}(k_1, \ldots, k_w)$ the subgraph of the circulant graph $\mathbb{Z}_{2t}(k_1, \ldots, k_w)$ with edge set $\{\{2v, 2v + k_j\} : 0 \leq v \leq t - 1,\ 1 \leq j \leq w\}$ where the sum is taken module $2t$. Moreover, we denote by $S_\infty(k_1, \ldots, k_w)$ the (infinite) graph when $\mathbb{Z}_{2t} = \mathbb{Z}$.

In the following lemma we describe some relevant properties of this graph:

Lemma 4.1 *Given an integer $t \geq 5$, and a sequence $k_1, \ldots, k_w$ of different odd elements from $\mathbb{Z}_{2t}$, the graph $S_{2t}(k_1, \ldots, k_w)$ is w-regular, bipartite and has at most w Cayley colors in $\mathbb{Z}_{2t}$. Moreover, the girth of $S_{2t}(k_1, \ldots, k_w)$ is at least 6 iff all the numbers $k_i - k_j$ are different for $i \neq j$ and $1 \leq i, j \leq w$. These properties hold for $2t = \infty$.*

Proof Given an odd element $k_j \in \mathbb{Z}_{2t}$, the set of edges $\{\{2v, 2v + k_j\} : 0 \leq v \leq t - 1\}$ defines a matching between the vertices with even label and the ones with odd label in $\mathbb{Z}_{2t}$. Therefore, $G = S_{2t}(k_1, \ldots, k_w)$ is w-regular, bipartite and has even girth $g \geq 4$.

Assume the numbers $k_i - k_j$ are different for $i \neq j$ and $1 \leq i, j \leq w$. We prove that the girth of G is greater or equal than 6. Suppose that there exists a 4-cycle $v_0 v_1 v_2 v_3 v_0$. By reordering, we may take v_0, v_2 even and v_1, v_3 odd. So, $v_1 = v_0 + k_i$, $v_2 = v_1 - k_j$, $v_3 = v_2 + k_p$, $v_0 = v_3 - k_q$ with $i \neq j$, $p \neq q, p \neq j, q \neq i$. Then, $k_i - k_j + k_p - k_q = 0$ and $k_i - k_j = k_q - k_p$ in $\mathbb{Z}_{2t}$ which is a contradiction, since by hypothesis all these numbers are different. Hence the girth of G must be at least 6 because it is bipartite. The proof is the same when $\mathbb{Z}_{2t} = \mathbb{Z}$, taking into account that in this case the equalities are considered in $\mathbb{Z}$. ■

The $(q + 1, 6)$-cages, with q a prime power, are known examples of this type of graphs. For instance, the Heawood graph can be constructed as $S_{14}(1, -1, 5)$. They can also be represented by using *perfect difference sets* (see [20, 28]) and as the Levi graphs of the projective plane over the field $\mathbb{F}_q$.

We can use a graph $S_{2t}(k_1, \ldots, k_w)$ with girth at least 6 to construct a new regular graph with the same order, greater degree and girth at least five.

Definition 4.1 *Given an integer $t \geq 5$, a sequence of different odd elements $k_1, \ldots, k_w$ and two different even elements $0 < P, Q < t$ from $\mathbb{Z}_{2t}$, we denote by $S_{2t}(P, Q; k_1, \ldots, k_w)$ the graph obtained adding to $S_{2t}(k_1, \ldots, k_w)$ the new edges $\{2v, 2v + P\}$ and $\{2v + 1, 2v + 1 + Q\}$, where the sum is taken modulo 2t. The graph $S_\infty(P, Q; k_1, \ldots, k_w)$ is defined in a similar way over $\mathbb{Z}$.*

Notice that if P divides $2t$ the subgraph of $S_{2t}(P, Q; k_1, \ldots, k_w)$ induced by the even numbers, is formed by $P/2$ cycles, each of them with size $2t/P$. Similar result holds when Q divides $2t$ and the subgraph of $S_{2t}(P, Q; k_1, \ldots, k_w)$ induced by the odd numbers. The standard Generalized Petersen Graphs with $2t$ vertices introduced by Coxeter in [12] are obtained as $S_{2t}(2, Q; 1)$ and the I-graph $I(t, j, k)$ in [34] as $S_{2t}(2j, 2k; 1)$. Funk uses in [17] a 4-regular generalization $P(k, \eta, \nu)$ of the Petersen graph which corresponds to $S_{2k}(2, 2\eta; 1, 2\nu + 1)$. As illustration, Figure 4 (left) depicts $S_{24}(2, 10; 1, 7)$ where the highlighted edges have weights 1, 2, 10, and Figure 4 (right) shows the $(5, 5)$-cage or Foster Graph, which corresponds to $S_{30}(6, 12; 1, -1, 9)$, where the three Petersen subgraphs contained in this cage are highlighted.

Next, we summarize some useful properties of these graphs.

Lemma 4.2 *The graph $S_{2t}(P, Q; k_1, \ldots, k_w)$, defined over $\mathbb{Z}_{2t}$, is $(w + 2)$-regular and has at most $w + 2$ Cayley colors. Moreover, the girth of $S_{2t}(P, Q; k_1, \ldots, k_w)$ is at least 5 if and only if the following conditions hold:*

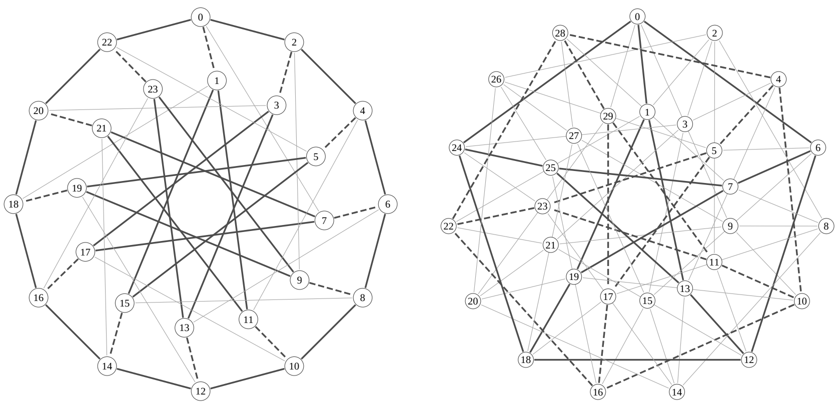

Figure 4: The graph $S_{24}(2, 10; 1, 7)$ and the $(5, 5)$-cage of Foster.

(i) *The numbers* $3P, 4P, 3Q, 4Q$ *are different from* 0 *in* $\mathbb{Z}_{2t}$.

(ii) *All the numbers* $k_i - k_j$ *are different for* $i \neq j$ *and* $1 \leq i, j \leq w$.

(iii) *No relation* $k_i - k_j = \omega - \omega'$ *holds, for a pair* $\omega, \omega' \in \Omega = \{0, \pm P, \pm Q\}$.

The result also holds when $Z_{2t} = Z$.

Proof Denote $G = S_{2t}(P, Q; k_1, \ldots, k_w)$. According to Lemma 4.1, the subgraph $B = S_{2t}(k_1, \ldots, k_w)$ is an w-regular bipartite graph with girth at least 6 iff item (ii) is satisfied. The partite sets of B are the set of even vertices, denoted by Ev, and the set of odd vertices, denoted by Od, of $\mathbb{Z}_{2t}$. Consider T_0 and T_1 the circulant graphs whose vertices are Ev and Od respectively, and whose edges are $\{2v, 2v + P\}$ and $\{2v + 1, 2v + 1 + Q\}$, respectively. Clearly, T_0 and T_1 are 2-regular and condition (i) that $3P, 4P, 3Q, 4Q \neq 0$ means that the subgraphs T_0 and T_1 have girth at least five. Now, observe that the graph G is an amalgamation $B(T_0, T_1)$ obtained by adding to Ev all the edges of T_0 and by adding to Od all the edges of T_1. Hence $G = S_{2t}(P, Q; k_1, \ldots, k_w)$ is $(w + 2)$-regular. Next, suppose that C is a cycle in G of size 3 or 4 which must contain even and odd vertices. If C has a single even vertex, we have either $k_i \pm Q - k_j = 0$ or $k_i \pm 2Q - k_j = 0$, depending on the size of C, and both equalities contradict (iii). If C contains two even and two odd vertices, we have $k_i \pm Q - k_j \pm P = 0$, again contradicting (iii). Therefore G has girth at least 5 iff conditions $(i), (ii), (iii)$ are satisfied. ■

Notice that it is useful to take $Q = 2P$ because in this case there are only four differences $\pm\{P, 2P, 3P, 4P\}$ to be avoided. Furthermore, if $S_{2t}(P, Q; k_1, \ldots, k_w)$ has girth $g \geq 5$, the (infinite) graph $S_\infty(P, Q; k_1, \ldots, k_w)$ also satisfies $g \geq 5$. We are interested in the converse result.

Definition 4.2 *We call span* D *of a graph* $S_\infty(P, Q; k_1, \ldots, k_w)$ *the maximum element of the set* $\{|k_i|, k_i - k_j, \omega - \omega'\}$, *with* $\omega, \omega' \in \{0, \pm P, \pm Q\}$.

Lemma 4.3 *Given even positive $P \neq Q$ and odd different $k_1, \ldots, k_w$ integers, let us consider a graph $S_\infty(P, Q; k_1, \ldots, k_w)$ with girth $g \geq 5$ and span D. If $t \geq D + 1$, then*

(*i*) *$0 < P, Q, |k_i| < t$, so the graph $S_{2t}(P, Q; k_1, \ldots, k_w)$ has regularity $w + 2$.*

(*ii*) *$S_{2t}(P, Q; k_1, \ldots, k_w)$ has girth at least 5.*

Proof By definition, $0 < P, Q \leq D$ and $-D \leq k_i \leq D$. As $t \geq D+1$, item (i) is immediate. Let also see that $S_{2t}(P, Q; k_1, \ldots, k_w)$ has girth $g \geq 5$. Given two different pairs of odd weights, we have $k_i - k_j \neq k_p - k_q$ in $\mathbb{Z}$. Also, from the definition of D, we have $-D \leq k_i - k_j, k_p - k_q \leq D$ and hence, $-2t < (k_i - k_j) - (k_p - k_q) < 2t$. So, $k_i - k_j \neq k_p - k_q$ in $\mathbb{Z}_{2t}$. The same argument shows $k_i - k_j \neq \omega - \omega'$ in $\mathbb{Z}_{2t}$. These are the conditions $(ii), (iii)$ of Lemma 4.2. Notice that condition (i) of the Lemma 4.2 has been explicitly stated. ■

As an example, let us mention that the graph $S_\infty(2, 4; 3, -7)$ has girth 5 and span $D = 10$. Therefore, the graph $S_{2t}(2, 4; 3, -7)$ is a 4-regular with girth 5 for orders $2t \geq 22$.

In the next subsection we construct two pairs of regular graphs of girth 5 suitable for amalgamation into L_q for some values of q.

4.2 Elliptic semiplanes of type L

Recall that the Levi Graph of an elliptic semiplane of type L is denoted by L_q. Following the terminology of Funk in [17] we say that two r-regular graphs G_0 and G_1 with girth at least five are *suitable for amalgamation into the elliptic semiplane L_q* if they are labeled with the elements of the cyclic group $(\mathbb{Z}_{q-1}, +)$ with disjoint sets of Cayley colors in this group. When q is odd, the fact that $\mathbb{Z}_{q-1}$ has $q - 1$ elements suggests the use of this semiplane, because r-regular graphs with odd degree have even order.

As in Section 3, the amalgamation of a pair of r-regular suitable graphs into the elliptic semiplane L_q gives a $(q + r, 5)$-graph $\mathcal{L}_q(G_0, G_1)$. It has $2(q^2 - 1)$ vertices and deleting pairs of blocks of vertices from $\mathcal{L}_q(G_0, G_1)$, for regularities $k \leq q + r$, we have

$$n(k, 5) \leq 2(q - 1)(k - r + 1). \tag{2}$$

With $q = 19$ vertices it is possible to construct a unique 4-regular graph with girth 5, the $(4, 5)$-cage due to Robertson in [24]. The use of the highest value of $r \geq 4$ for a given $q > 19$ increases the accuracy of the inequality (2). Funk in [17] constructs the best possible regular amalgams for $q \in \{23, 25, 27\}$ and hence we focus on primes $q \geq 29$. Next, we give a construction of graphs which provide accurate amalgams for $q \in \{29, 31, 37, 41, 43, 47\}$.

Construction 2:

- For $q = 29$:

Consider the graphs $G_0 = S_{28}(4, 8; 1, -1)$ and $G_1 = S_{28}(2, 6; 3, -7)$ showed in Figure 5. They are a suitable pair, that is, both graphs are 4-regular, have girth five and have disjoint sets of Cayley colors, concretely $\pm\{1, 4, 8\}$ and $\pm\{2, 3, 6, 7\}$, respectively. Hence, the 33-regular graph $\mathcal{L}_{29}(G_0, G_1)$ has girth 5 and order 1680. We have generated the graph $\mathcal{L}_{29}(G_0, G_1)$ and observed that it has diameter 4. Deletion and inequality (2) provide

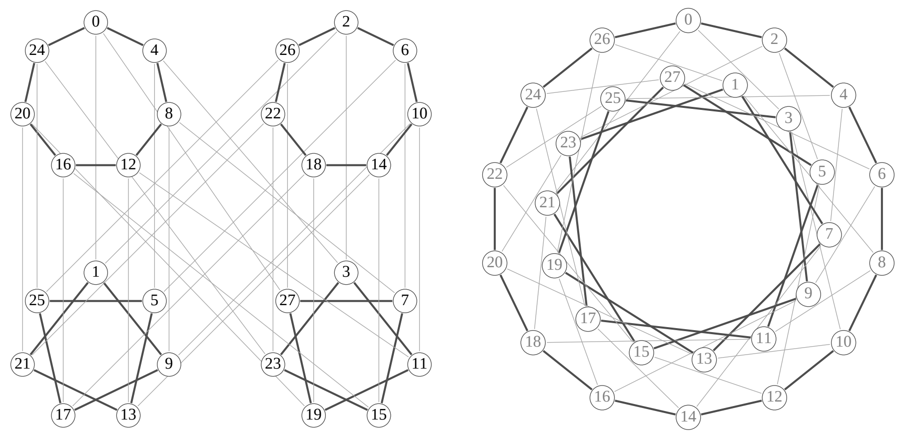

Figure 5: $S_{28}(4, 8; 1, -1)$ and $S_{28}(2, 6; 3, -7)$, a pair of suitable graphs over $\mathbb{Z}_{28}$.

entries $k = 32, 33$ of Table 2.

- For $q = 31$:

  There exist four (5,5)-cages (see [22, 25, 29, 31, 33]) and the graph $G_0 = S_{30}(6, 12; 1, -1, 9)$ is isomorphic to the Foster one. The second suitable half G_1 has been found with the following relabeling of the vertices.

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$0 \leftrightarrow 0$	$1 \leftrightarrow 28$	$2 \leftrightarrow 1$	$3 \leftrightarrow 27$	$4 \leftrightarrow 2$	$5 \leftrightarrow 19$
$6 \leftrightarrow 4$	$7 \leftrightarrow 7$	$8 \leftrightarrow 5$	$9 \leftrightarrow 22$	$10 \leftrightarrow 6$	$11 \leftrightarrow 3$
$12 \leftrightarrow 8$	$13 \leftrightarrow 24$	$14 \leftrightarrow 9$	$15 \leftrightarrow 20$	$16 \leftrightarrow 10$	$17 \leftrightarrow 15$
$18 \leftrightarrow 12$	$19 \leftrightarrow 23$	$20 \leftrightarrow 13$	$21 \leftrightarrow 11$	$22 \leftrightarrow 14$	$23 \leftrightarrow 29$
$24 \leftrightarrow 16$	$25 \leftrightarrow 21$	$26 \leftrightarrow 17$	$27 \leftrightarrow 25$	$28 \leftrightarrow 18$	$29 \leftrightarrow 26$

  Since the Cayley colors of G_1 are the elements of the set $\mathbb{Z}_{30} - \{0, \pm 1, \pm 6, \pm 9, \pm 12\}$, the graphs G_0 and G_1 have disjoint Cayley colors, and therefore, the amalgam graph $\mathcal{L}_{31}(G_0, G_1)$ has girth 5, regularity 36 and order $2(31^2 - 1) = 1920$. Block deletion provides $n(35, 5) \leq 1860$ and $n(34, 5) \leq 1800$.

- For $q = 37$:

  Consider the graphs $G_0 = S_{36}(8, 14; 1, -1, 11)$ and $G_1 = S_{36}(2, 4; 3, -7, 15)$ defined on the cyclic group $(\mathbb{Z}_{36}, +)$. Both graphs are 5-regular, have girth five and disjoint Cayley colors, concretely $\pm\{1, 8, 11, 14\}$ and $\pm\{2, 3, 4, 7, 15\}$, respectively. Hence, the 42-regular graph $\mathcal{L}_{37}(G_0, G_1)$ has girth 5 and order 2736. Deletion provides $n(41, 5) \leq 2664$, $n(40, 5) \leq 2592$, $n(39, 5) \leq 2520$, $n(38, 5) \leq 2448$.

- For $q = 41$:

  The $(6,5)$-cage is unique and it is well known (see [23]) that it can be constructed by removing the vertices of a Petersen graph from the Hoffman-Singleton cage. We present a construction of the $(6,5)$-cage as the graph $S_{40}(8,16;1,-1,5,-13)$. We denote it by G_0. Due to the uniqueness of this cage, we construct a suitable graph G_1 according to the following relabeling of the vertices.

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$0 \leftrightarrow 0$	$1 \leftrightarrow 12$	$2 \leftrightarrow 1$	$3 \leftrightarrow 20$	$4 \leftrightarrow 2$	$5 \leftrightarrow 33$	$6 \leftrightarrow 3$	$7 \leftrightarrow 37$
$8 \leftrightarrow 7$	$9 \leftrightarrow 38$	$10 \leftrightarrow 8$	$11 \leftrightarrow 18$	$12 \leftrightarrow 9$	$13 \leftrightarrow 32$	$14 \leftrightarrow 10$	$15 \leftrightarrow 25$
$16 \leftrightarrow 14$	$17 \leftrightarrow 5$	$18 \leftrightarrow 15$	$19 \leftrightarrow 13$	$20 \leftrightarrow 16$	$21 \leftrightarrow 36$	$22 \leftrightarrow 17$	$23 \leftrightarrow 27$
$24 \leftrightarrow 21$	$25 \leftrightarrow 19$	$26 \leftrightarrow 22$	$27 \leftrightarrow 11$	$28 \leftrightarrow 23$	$29 \leftrightarrow 35$	$30 \leftrightarrow 24$	$31 \leftrightarrow 39$
$32 \leftrightarrow 28$	$33 \leftrightarrow 26$	$34 \leftrightarrow 29$	$35 \leftrightarrow 4$	$36 \leftrightarrow 30$	$37 \leftrightarrow 34$	$38 \leftrightarrow 31$	$39 \leftrightarrow 6$

  Since G_0 and G_1 have no Cayley color in common, the 47-regular graph $\mathcal{L}_{41}(G_0, G_1)$ has girth 5 and order $2(41^2-1) = 3360$. Deletion and inequality (2) provide entries $k = 43 \ldots 46$ of Table 2.

- For $q = 43, 47$:

  Since we resort to 5-regularity, there exist several pairs of suitable graphs. For the sake of uniformity, we consider the graphs $S_{q-1}(6,12;1,-1,9)$ and $S_{q-1}(2,4;3,-7,15)$. It proves that $n(48,5) \leq 2(43^2-1) = 3696$ and $n(52,5) \leq 2(47^2-1) = 4416$. As a curiosity, let us mention that the graph $S_{42}(1,-1,-7,11,15)$ is the $(5,6)$-cage and it forms a suitable pair with $S_{42}(2,4;5,-5,17)$.

Based on the above constructions and recalling that it is possible to delete blocks of points and lines we can write the following theorem.

Theorem 4.1 *The following upper bound on the order of a k-regular graph of girth 5 holds*

k	$rec(k,5)$
$32, 33$	$56(k-3)$
$34, 35, 36$	$60(k-4)$
$38, \ldots, 42$	$72(k-4)$
$43, \ldots, 47$	*80 (k-5)*
48	3696
$49, \ldots, 52$	$92(k-4)$

To finalize this section we prove Theorem 1.1. In this case we generate a pair of 6-regular suitable graphs to be amalgamated into L_q, for an odd prime power $q \geq 49$. We start with $q = 49$; notice that this case is sharp because the Hoffman-Singleton graph is the cage that attains the lower bound $n_0(7,5) = 50$ (see [19])

Theorem 1.1 *Given an integer $k \geq 53$, let q be the lowest odd prime power, such that $k \leq q+6$. Then $n(k,5) \leq 2(q-1)(k-5)$.*

Proof First consider $q = 49$. Add to the 4-regular bipartite graph $S_{48}(1, -1, 5, -13)$ the edges $\{2v, 2v + 8\}$ over the even vertices of $\mathbb{Z}_{48}$, and the four cycles $\{1 + i, 17 + i, 41 + i, 25 + i, 9 + i, 33+ i, 1+i\}$, for $i = 0, 2, 4, 6$, over the odd vertices. We call G_0 to this $(6, 5)$-graph. To construct a suitable graph G_1, we resort to the following relabeling of the vertices

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$0 \leftrightarrow 0$	$1 \leftrightarrow 42$	$2 \leftrightarrow 1$	$3 \leftrightarrow 39$	$4 \leftrightarrow 2$	$5 \leftrightarrow 23$	$6 \leftrightarrow 3$	$7 \leftrightarrow 47$
$8 \leftrightarrow 6$	$9 \leftrightarrow 4$	$10 \leftrightarrow 7$	$11 \leftrightarrow 28$	$12 \leftrightarrow 8$	$13 \leftrightarrow 34$	$14 \leftrightarrow 9$	$15 \leftrightarrow 43$
$16 \leftrightarrow 12$	$17 \leftrightarrow 35$	$18 \leftrightarrow 13$	$19 \leftrightarrow 36$	$20 \leftrightarrow 14$	$21 \leftrightarrow 29$	$22 \leftrightarrow 15$	$23 \leftrightarrow 44$
$24 \leftrightarrow 18$	$25 \leftrightarrow 37$	$26 \leftrightarrow 19$	$27 \leftrightarrow 5$	$28 \leftrightarrow 20$	$29 \leftrightarrow 40$	$30 \leftrightarrow 21$	$31 \leftrightarrow 10$
$32 \leftrightarrow 24$	$33 \leftrightarrow 45$	$34 \leftrightarrow 25$	$35 \leftrightarrow 46$	$36 \leftrightarrow 26$	$37 \leftrightarrow 38$	$38 \leftrightarrow 27$	$39 \leftrightarrow 16$
$40 \leftrightarrow 30$	$41 \leftrightarrow 41$	$42 \leftrightarrow 31$	$43 \leftrightarrow 17$	$44 \leftrightarrow 32$	$45 \leftrightarrow 11$	$46 \leftrightarrow 33$	$47 \leftrightarrow 22$

The graphs G_0 and G_1 have disjoint Cayley colors, namely $w(G_0) = \pm\{1, 5, 8, 13, 16, 24\}$ and $w(G_1) = \mathbb{Z}_{48} - (w(G_0) \cup \{0\})$. Hence, G_0 and G_1 is a suitable pair of graphs for amalgamation into L_{49}. Using these graphs and also the fact that we can delete blocks of points and lines we prove the theorem for $53 \leq k \leq 55$.

When $q \in \{53, 67, 71, 79, \ldots\}$ is an odd prime power, we consider the 6-regular graphs $G_0 = S_{q-1}(8, 16; 1, -1, 5, -13)$ and $G_1 = S_{q-1}(2, 4; 3, -7, 15, -21)$. Direct checking shows their suitability over L_q for $q = 53, 67, 71$. When $q \geq 79$, the suitability of G_0 and G_1 is a consequence of Lemma 4.3, because the infinite graphs $S_\infty(8, 16; 1, -1, 5, -13)$ and $S_\infty(2, 4; 3, -7, 15, -21)$ have girth 5 and spans 32 and 37, respectively. When $q \in \{59, 61, 73\}$, the graph $G_0 = S_{q-1}(8, 16; 1, -1, 5, -13)$ combined with $G_1 = S_{q-1}(2, 4; 3, -7, 15, \alpha)$, where $\alpha = -23$ for $q = 59, 73$ and $\alpha = -25$ for $q = 61$, is a suitable pair of graphs over L_q. Therefore, for $q \geq 49$, the $(q+6)$-regular graph $\mathcal{L}_q(G_0, G_1)$ has girth at least 5 and order $2(q^2 - 1)$. Also, according to inequality (2), $n(k, 5) \leq 2(q - 1)(k - 5)$, for regularities $56 \leq k \leq q + 6$. ■

Notice that Theorem 1.1 improves Jørgensen's result $n(q + \lfloor \frac{\sqrt{q-1}}{4} \rfloor, 5) \leq 2(q^2 - 1)$ (see [20]) for $k \leq 577$ and ties with it for $578 \leq k \leq 779$.

5 General constructions for $q = 2^m$.

In this section we work with the same ideas used in the two previous sections. We amalgamate into C_q for $q = 2^m$ when $m \geq 5$ applying Theorem 3.1 on regular graphs. The case $m = 4$ is considered in Section 3, where we amalgamate bi-regular graphs.

First, we deal with $m = 5$ or $q = 32$. Since an r-regular graph with 32 vertices and girth 5 can reach at most 5-regularity, we have the following sharp result:

Theorem 5.1 *There exists a* 37*-regular graph with girth* 5 *and order* 2048.

Proof As in the case $q = 16$, denote the elements of $(\mathbb{F}_{32}, +) \cong ((\mathbb{Z}_2)^5, +)$ by $defgh$ instead of $\{d, e, f, g, h\}$. Let G_0 be the $(5, 5)$-graph with order 32 and with the following adjacency list:

Vertex	Adjacent vertices		Vertex	Adjacent vertices
00000	10000, 11010, 11100, 00001, 11111		00001	00000, 10001, 11011, 11101, 11110
10000	00000, 01011, 01101, 01110, 11001		10001	00001, 01010, 01100, 01111, 11000
01000	01001, 10010, 10101, 10110, 11000		01001	01000, 10011, 10100, 10111, 11001
11000	00011, 00100, 00111, 01000, 10001		11001	00010, 00101, 00110, 01001, 10000
00100	00101, 10100, 11000, 11010, 11110		00101	00100, 10101, 11001, 11011, 11111
10100	00100, 01001, 01011, 01111, 11101		10101	00101, 01000, 01010, 01110, 11100
01100	01101, 10001, 10011, 10110, 11100		01101	01100, 10000, 10010, 10111, 11101
11100	00000, 00010, 00111, 01100, 10101		11101	00001, 00011, 00110, 01101, 10100
00010	00011, 10010, 11001, 11100, 11110		00011	00010, 10011, 11000, 11101, 11111
10010	00010, 01000, 01101, 01111, 11011		10011	00011, 01001, 01100, 01110, 11010
01010	01011, 10001, 10101, 10111, 11010		01011	01010, 10000, 10100, 10110, 11011
11010	00000, 00100, 00110, 01010, 10011		11011	00001, 00101, 00111, 01011, 10010
00110	00111, 10110, 11001, 11010, 11101		00111	00110, 10111, 11000, 11011, 11100
10110	00110, 01000, 01011, 01100, 11111		10111	00111, 01001, 01010, 01101, 11110
01110	01111, 10000, 10011, 10101, 11110		01111	01110, 10001, 10010, 10100, 11111
11110	00001, 00010, 00100, 01110, 10111		11111	00000, 00011, 00101, 01111, 10110

The set $w(G_0) = \{00001, 01001, 10000, 11010, 11011, 11100, 11101, 11110, 11111\}$ contains the Cayley colors of G_0. As graph G_1, consider the isomorphic graph of G_0 with the following relabeling of the vertices:

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$00000 \leftrightarrow 00000$	$00001 \leftrightarrow 00011$	$00010 \leftrightarrow 00010$	$00011 \leftrightarrow 00001$	$00100 \leftrightarrow 00100$	$00101 \leftrightarrow 00111$
$00110 \leftrightarrow 00110$	$00111 \leftrightarrow 01110$	$01000 \leftrightarrow 11001$	$01001 \leftrightarrow 11100$	$01010 \leftrightarrow 11111$	$01011 \leftrightarrow 11011$
$01100 \leftrightarrow 10011$	$01101 \leftrightarrow 11101$	$01110 \leftrightarrow 11010$	$01111 \leftrightarrow 11110$	$10000 \leftrightarrow 01111$	$10001 \leftrightarrow 10100$
$10010 \leftrightarrow 01100$	$10011 \leftrightarrow 10000$	$10100 \leftrightarrow 01000$	$10101 \leftrightarrow 10001$	$10110 \leftrightarrow 01010$	$10111 \leftrightarrow 11000$
$11000 \leftrightarrow 10110$	$11001 \leftrightarrow 01101$	$11010 \leftrightarrow 10101$	$11011 \leftrightarrow 01001$	$11100 \leftrightarrow 00101$	$11101 \leftrightarrow 01011$
$11110 \leftrightarrow 10111$	$11111 \leftrightarrow 10010$				

Since the set of Cayley colors of G_1 is $w(G_1) = \mathbb{F}_{32} - (w(G_0) \cup \{0000, 00110\})$, the graphs G_0 and G_1 have disjoint Cayley colors, and therefore, the amalgam graph $\mathcal{C}_{32}(G_0, G_1)$ has girth 5, regularity 37 and order $2 \cdot 32^2 = 2048$. ■

To give a general result for $m \geq 6$ we need some equivalences and definitions. As usual we identify the elements of $\mathbb{F}_{2^m} \cong (\mathbb{Z}_2)^m$ with a number of $\mathbb{Z}_{2^m}$ in the following way:

$$(v_{m-1}, \dots, v_0) \longleftrightarrow \sum_{i=0}^{m-1} 2^i v_i$$

for every $i = 0, \dots, m-1$ and $v_i \in \mathbb{Z}_2$. This induces a bijection $\phi : \mathbb{Z}_{2^m} \to (\mathbb{Z}_2)^m$ such that the elements of $(\mathbb{Z}_2)^m$ can be represented either by a vector or by a number.

This bijective relationship allows to translate the graph $S_{2^m}(P, Q; k_1, \dots, k_w)$ with vertex set $\mathbb{Z}_{2^m}$ into a new graph with vertex set $(\mathbb{Z}_2)^m$ defined as follows:

Definition 5.1 *Given an integer $N = 2^m$, a sequence $k_1, \dots, k_w$ of different odd elements from $\mathbb{Z}_N$ and two even elements $0 < P, Q < N/2$, we denote by $\bar{S}_{2^m}(P, Q; k_1, \dots, k_w)$ the graph with vertex set $(\mathbb{Z}_2)^m$ obtained by translating the vertices and edges of $S_{2^m}(P, Q; k_1, \dots, k_w)$ by means of the bijection $\phi : \mathbb{Z}_{2^m} \to (\mathbb{Z}_2)^m$.*

Clearly, graphs $S_{2^m}(P,Q;k_1,\dots,k_w)$ and $\bar{S}_{2^m}(P,Q;k_1,\dots,k_w)$ are isomorphic. Notice that the Cayley colors of the graph $\bar{S}_{2^m}(P,Q;k_1,\dots,k_w)$ are computed in the additive group $(\mathbb{Z}_2)^m$; which implies that edges of $\bar{S}_{2^m}(P,Q;k_1,\dots,k_w)$ associated to an element of $\{P,Q;k_1,\dots,k_w\}$ might have different Cayley colors in $(\mathbb{Z}_2)^m$.

To finish this section we prove Theorem 1.2, in which we consider even primes $q \geq 64$ and construct a pair of suitable 6-regular graphs whose amalgamation into C_q establishes a general bound on $n(k,5)$ for regularities $68 \leq k \leq q+6$.

Theorem 1.2 *Given an integer $k \geq 68$, let $q = 2^m$ be the lowest even prime, such that $k \leq q+6$. Then $n(k,5) \leq 2q(k-6)$.*

Proof Consider $q = 2^m$ for an integer $m \geq 6$. Due to the bijection ϕ described above we represent the elements of $(\mathbb{Z}_2)^m$ by the numbers of $\mathbb{Z}_{2^m}$ and vice versa.

For $q = 64$ we consider the 6-regular graph $G_0 = \bar{S}_{64}(4,8;1,3,41,47)$ of girth five and set of Cayley colors $w(G_0) = \{1,3,4,7,8,12,15,19,23,24,25,28,31,41,47,51,55,56,57,60,63\}$. To obtain the graph G_1 we consider the following relabeling of the vertices:

$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$	$G_0 \leftrightarrow G_1$
$0 \leftrightarrow 0$	$1 \leftrightarrow 44$	$2 \leftrightarrow 2$	$3 \leftrightarrow 39$	$4 \leftrightarrow 5$	$5 \leftrightarrow 41$	$6 \leftrightarrow 7$	$7 \leftrightarrow 19$
$8 \leftrightarrow 12$	$9 \leftrightarrow 50$	$10 \leftrightarrow 14$	$11 \leftrightarrow 28$	$12 \leftrightarrow 1$	$13 \leftrightarrow 52$	$14 \leftrightarrow 3$	$15 \leftrightarrow 21$
$16 \leftrightarrow 4$	$17 \leftrightarrow 25$	$18 \leftrightarrow 6$	$19 \leftrightarrow 22$	$20 \leftrightarrow 57$	$21 \leftrightarrow 20$	$22 \leftrightarrow 59$	$23 \leftrightarrow 31$
$24 \leftrightarrow 24$	$25 \leftrightarrow 45$	$26 \leftrightarrow 26$	$27 \leftrightarrow 56$	$28 \leftrightarrow 61$	$29 \leftrightarrow 48$	$30 \leftrightarrow 63$	$31 \leftrightarrow 29$
$32 \leftrightarrow 32$	$33 \leftrightarrow 10$	$34 \leftrightarrow 34$	$35 \leftrightarrow 8$	$36 \leftrightarrow 49$	$37 \leftrightarrow 23$	$38 \leftrightarrow 51$	$39 \leftrightarrow 27$
$40 \leftrightarrow 36$	$41 \leftrightarrow 62$	$42 \leftrightarrow 38$	$43 \leftrightarrow 54$	$44 \leftrightarrow 9$	$45 \leftrightarrow 35$	$46 \leftrightarrow 11$	$47 \leftrightarrow 43$
$48 \leftrightarrow 40$	$49 \leftrightarrow 46$	$50 \leftrightarrow 42$	$51 \leftrightarrow 30$	$52 \leftrightarrow 53$	$53 \leftrightarrow 33$	$54 \leftrightarrow 55$	$55 \leftrightarrow 17$
$56 \leftrightarrow 16$	$57 \leftrightarrow 58$	$58 \leftrightarrow 18$	$59 \leftrightarrow 60$	$60 \leftrightarrow 13$	$61 \leftrightarrow 47$	$62 \leftrightarrow 15$	$63 \leftrightarrow 37$

The Cayley colors of G_1 are $w(G_1) = \{1,\dots,63\} - w(G_0) - \{50\}$ and hence the $(70,5)$-graph $\mathcal{C}_{64}(G_0,G_1)$ has order $2 \cdot 64^2$.

In general for $q = 2^m$ and $m \geq 7$ we use the previous graphs G_0 and G_1 defined over $(\mathbb{Z}_2)^6$ to construct new graphs G_0^m and G_1^m with vertex set $(\mathbb{Z}_2)^m$ in the following way: The neighbors of a vertex $(u_{m-1},\dots,u_0)$ in G_0^m are the six vertices of the set $\{(u_{m-1},...,u_6,v_5,\dots,v_0) : (u_5,\dots,u_0)(v_5,\dots,v_0) \in E(G_0)\}$. Similar definition holds for G_1^m. Graphs G_0^m and G_1^m are formed by 2^{m-6} disconnected copies of G_0 and G_1, respectively, and therefore, both graphs are 6-regular with girth 5. Also, the sets of Cayley colors $w(G_0^m) = \{(0,\dots,0,\alpha_5,\dots,\alpha_0) \in (\mathbb{Z}_2)^m : (\alpha_5,\dots,\alpha_0) \in w(G_0)\}$ and $w(G_2^m) = \{(0,\dots,0,\beta_5,\dots,\beta_0) \in (\mathbb{Z}_2)^m : (\beta_5,\dots,\beta_0) \in w(G_1)\}$ are disjoint because $w(G_0) \cap w(G_1) = \emptyset$. Clearly, the graphs G_0^m and G_1^m are suitable for amalgamation into $\mathcal{C}_q$ and the graph $\mathcal{C}_q(G_0^m, G_1^m)$ has regularity $q+6$, order $2q^2$ and girth at least five. For $k \leq q+6$ removing $q+6-k$ blocks of points and $q+6-k$ blocks of lines we obtain a graph of order $2q^2 - 2q(q+6-k)$ and consequently $n(k,5) \leq 2q(k-6)$. ■

Clearly in this paper we improve $rec(k,5)$ for many values of k. As we mention at the end of Section 4 our Theorem 1.1 improves Jørgensen's result for $k \leq 577$. We consider that an interesting future work would be to extend Theorem 1.1 to large odd prime powers and to improve our Theorem 1.2 when q is a power of two.

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